$$\varepsilon^* = \frac{1}{m_1+1}\left(\frac{\sigma_0}{C}\right)^{\frac{m_1+1}{n_1+2}} + \frac{\sigma_0}{DL_1}\left[\left(\frac{\sigma_0}{C}\right)^{\frac{L_1}{n_1+2}} - s_0^{L_1}\right] - \frac{C}{D}\ln\left[\frac{1}{s_0}\left(\frac{\sigma_0}{C}\right)^{\frac{1}{n_1+2}}\right].$$

Presented in the figure as an illustration are the dependences $\varepsilon_0(\sigma_0)$ and $\varepsilon^*(\sigma_0)$ for $C = D = 1$, $n_1 = 2$, $m_1 = 6$. We examine two combinations of values of the remaining exponents characterized by the parameter k. In the case $k = 1$ the exponents m_2 and n_2 take the values $m_2 = 9$ and $n_2 = 7$, while in the case $k = 2$ they are $m_2 = 7$ and $n_2 = 9$. The curve $\varepsilon_0(\sigma_0)$ which is common to both cases is superposed by solid lines, while the curves $\varepsilon^*(\sigma_0)$ for $k = 1$ and 2 are superposed, respectively, by dashed and dash-dot lines. It is evident from (17) and Fig. 1 that the dependence $\varepsilon^*(\sigma_0)$ is not monotonic for $k = 1$ and decreases monotonically for $k = 2$.

Let us note that the nonmonotonicity of the function $\varepsilon^*(\sigma_0)$ results from the relationship of the exponents for components governing the creep characteristics of the material. The nonmonotonicity appears when the creep rate has a higher degree of dependence on the stress as compared with the dependence of the cumulative creep damage. In this case taking account of the nonlinear instantaneous characteristics plays a part analogous to that of the nonmonotonicity noted earlier when using different functional dependences for the creep rate and the cumulative damage rate [4].

LITERATURE CITED

1. Yu. N. Rabotnov, Creep of Structural Elements [in Russian], Nauka, Moscow (1966).
2. H. Broberg, "A new criterion for brittle creep rupture," Trans. ASME, E41, No. 3 (1974).
3. P. O. Boström, H. Broberg, L. Bräthe, and M. Chrzanowski, "On failure conditions in viscoelastic media and structures," Int. Symposium on Mechanics of Viscoelastic Media and Bodies, Springer-Verlag, Berlin (1975).
4. A. M. Lokoshchenko and S. A. Shesterikov, "Model of the creep strength with nonmonotonic dependence of the strain on the stress during rupture," Zh. Prikl. Mekh. Tekh. Fiz., No. 1 (1982).

STEADY-STATE CREEP IN REFRACTORY COMPOSITES

V. V. Dudukalenko and S. P. Shapovalov

UDC 539.376

There are anomalous variations in creep rate in the range 1000-1800°C for corundum-based refractories having high concentrations of ZrO_2 inclusions.

Measurements have been made [1] on torsion on cylindrical specimens under conditions of steady-state creep at various temperatures. The characteristic dependence of the creep rate on load (Fig. 1) shows that there is an anomalous viscosity change in the region of the phase transition temperature of ZrO_2 [1].

1. We use a form of the Bingham model to approximate the relationship shown in Fig. 1. The dissipative functions D are taken as differing from small and large strain rates ε_{ij} and for the corresponding stresses σ_{ij} [2]:

$$D = k\sqrt{\varepsilon_{ij}\,\varepsilon_{ij}} + (1/2)\nu\varepsilon_{ij}\varepsilon_{ij},\ s_{ij} = \sigma_{ij} - (1/3)\sigma_{ll}\,\delta_{ij} = \partial D/\partial\varepsilon_{ij},$$

where k is the plasticity limit and ν is the viscosity, while the subscript a in k_a and ν_a denotes those quantities for high strain rates.

The invariants $\gamma = \sqrt{\varepsilon_{ij}\varepsilon_{ij}}$, $\tau = \sqrt{s_{ij}s_{ij}}$ always simultaneously act as symbols for the shear components in torsion. In the (ρ, φ) polar coordinate system, the maximal stresses are obtained at the surface of a rod $\rho = R$, while the transition from $\tau = k + \nu\gamma$ to $\tau = k_a + \nu_a\gamma$ may occur at $\rho = R_1$. The zone of rapid creep propagates towards the center of the rod as the stresses increase. It can be shown that the piecewise-linear $\tau(\gamma)$ relationship goes over to a smooth conjugate one for the dependence of the torsional moment M on the torsion rate θ. As $\gamma = \theta\rho$, we have

---

Sumy. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 142-145, July-August, 1984. Original article submitted June 24, 1983.

0021-8944/84/2504-0635$08.50

$$M=2\pi\left[\int_{0}^{R_1}\tau\rho^2d\rho+\int_{R_1}^{R}\tau\rho^2d\rho\right],\quad M=2\pi\left[k\frac{R^3}{3}+\nu\theta\frac{R^4}{4}\right]\ \text{for}\ R_1>R; \tag{1.1}$$

$$M=2\pi\left[k_a\frac{R^3}{3}+\nu_a\theta\frac{R^4}{4}+\frac{(k_a-k)^4}{12\theta^3(\nu_a-\nu)^3}\right],\quad R_1=\frac{1}{\theta}\frac{k_a-k}{\nu-\nu_a}\leqslant R. \tag{1.2}$$

Then (1.1) and (1.2) relate the parameters of the approximating relationships (Fig. 1) to the parameters k, k_a, ν, ν_a of the generalized Bingham model. Formula (1.1) defines a linear dependence on θ, while (1.2) defines an asymptotically linear one ($\theta \to \infty$), which enables one to calculate the parameters of the piecewise-linear $\tau(\gamma)$ relationship from the experimental data (Fig. 1).

When an electrically fused refractory cools in a vitreous matrix containing ZrO_2 crystals, the viscosity increases because rigid corundum crystals are formed. Under these conditions, the material can experience high stresses, which may cause plastic strain in the ZrO_2 crystals, and the twinned structure in the latter explains the plasticity limit and the low viscosity of the inclusions.

If we use the above as a hypothesis, we can show that a two-phase structure whose components satisfy the Bingham relations is described by rheological relations characterizing the anomalous transition to high creep rates. This transition corresponds to the onset of plastic strain in the inclusions.

2. The two-phase structure consists of viscoplastic components that obey the Bingham rheological equations, which specify the relationship between the stresses σ_{ij} and the strain rates ε_{ij} in the form

$$s_{ij}=k\frac{\varepsilon_{ij}}{\sqrt{\varepsilon_{kl}\varepsilon_{kl}}}+\nu\varepsilon_{ij},\quad s_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{ll}\delta_{ij},$$

where k is the plasticity limit and ν is the viscosity of the matrix. The parameters of the inclusions are denoted by k_1 and ν_1.

In a macroscopic volume V consisting of regions V_1 for the inclusions and V_2 for the matrix, the dissipative function D is [3]

$$D=\frac{1}{V}\int_{V_2}\left(k\sqrt{\varepsilon_{ij}\varepsilon_{ij}}+\frac{1}{2}\nu\varepsilon_{ij}\varepsilon_{ij}\right)dV+\frac{1}{V}\int_{V_1}\left(k\sqrt{\varepsilon_{ij}\varepsilon_{ij}}+\frac{1}{2}\nu_1\varepsilon_{ij}\varepsilon_{ij}\right)dV. \tag{2.1}$$

In [3] we find a method of solving an analogous problem for the case of ideally plastic components in a heterogeneous structure. If one minimizes the function of (2.1) with the expected fluctuations for fixed mean values $\langle\varepsilon_{ij}\rangle$, one can calculate $D(\langle\varepsilon_{ij}\rangle)$, and therefore one can define the equations for the mean stresses $\langle\sigma_{ij}\rangle = \partial D/\partial\langle\varepsilon_{ij}\rangle$. We use formulas for averaging over regions V_1 and V_2 correspondingly for spherical inclusions: $\langle\varepsilon_{ij}\varepsilon_{ij}\rangle_1 = \langle\varepsilon_{ij}\rangle_1\langle\varepsilon_{ij}\rangle_1$, $\langle\sqrt{\varepsilon_{ij}\varepsilon_{ij}}\rangle_2 = \sqrt{\langle\varepsilon_{ij}\varepsilon_{ij}\rangle_2}$, which corresponds to the hypothesis of evaluating (2.1) for an optimally coupled matrix.

Let the geometrical structure of the material be defined by the isotropic function $\varkappa$, which takes the value 1 on the inclusions and 0 on the matrix. Clearly, $\langle\varkappa\rangle = c = V_1/V$, and we have

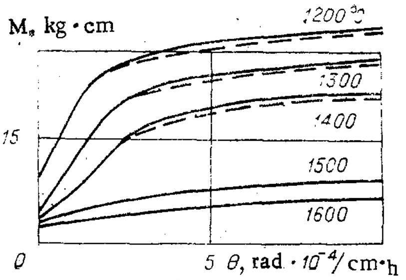

Fig. 1

$$(1-c)\langle\varepsilon_{ij}\varepsilon_{ij}\rangle_2=\langle\varepsilon_{ij}\varepsilon_{ij}\rangle-c\langle\varepsilon_{ij}\varepsilon_{ij}\rangle_1,\qquad(2.2)$$
$$\langle\varkappa'\varepsilon'_{ij}\rangle=c\left(\langle\varepsilon_{ij}\rangle_1-\langle\varepsilon_{ij}\rangle\right),$$

where the primes denote fluctuations.

We introduce the symbols

$$I_0^2=\langle\varepsilon_{ij}\rangle\langle\varepsilon_{ij}\rangle,\quad I_1^2=\langle\varepsilon_{ij}\rangle_1\langle\varepsilon_{ij}\rangle_1,$$
$$I^2=I_0^2+\langle\varepsilon'_{ij}\varepsilon'_{ij}\rangle-cI_1^2.$$

Then (2.1) is put as

$$D=k\sqrt{1-c}\,I+k_1cI_1+(1/2)\nu I^2+(1/2)\nu_1cI_1^2.\qquad(2.3)$$

The condition for a turning point in (2.3) leads to the equations

$$k\sqrt{1-c}\left(\varepsilon'_{ij,j}-\langle\varepsilon_{ij}\rangle_1\varkappa'_{,j}\right)/I+k_1\langle\varepsilon_{ij}\rangle\varkappa'_{,j}/I_2+\nu\varepsilon'_{ij,j}+(\nu_1-\nu)\langle\varepsilon_{ij}\rangle_1\varkappa'_{,j}=p'_{,i},\ \nu'_{i,i}=0,\qquad(2.4)$$

where p' represents the fluctuations in the hydrostatic pressure.

The solution to (2.4) is obtained as spectral expansions in terms of the wave numbers ξ_i [3]:

$$\varepsilon'_{ij}=\left(\langle\varepsilon_{il}\rangle_1\xi_l\xi_j\xi^{-2}+\langle\varepsilon_{jl}\rangle_1\xi_l\xi_i\xi^{-2}-2\langle\varepsilon_{kl}\rangle_1\xi_k\xi_l\xi_i\xi_j\right)A\varkappa',$$
$$\xi=\sqrt{\xi_i\xi_j},\ A=(\nu_1+k_1/I_1)/(\nu+k\sqrt{1-c/I}),\qquad(2.5)$$

and then because $\varkappa'$ is isotropic we have

$$\langle\varkappa'\varepsilon'_{ij}\rangle=c\left(\langle\varepsilon_{ij}\rangle_1-\langle\varepsilon_{ij}\rangle\right)=\frac{2}{5}c(1-c)A\langle\varepsilon_{ij}\rangle_1,\qquad(2.6)$$
$$\langle\varepsilon'_{ij}\varepsilon'_{ij}\rangle=\frac{2}{5}c(1-c)A^2\langle\varepsilon_{ij}\rangle_1\langle\varepsilon_{ij}\rangle_1.$$

Formulas (2.5) give a system of equations for the invariants I and I_1:

$$I_1\left(1-\frac{2}{5}(1-c)A\right)=I_0,\ I^2=I_0^2-cI_1^2+\frac{2}{5}c(1-c)A^2I_1^2.\qquad(2.7)$$

In accordance with the values $\langle\sigma_{ij}\rangle$ of the stresses, one can get deformation in which the inclusions behave as rigid particles. The latter begin to deform when critical stresses are reached. As $I_1=0$ for undeformable inclusions, we have from (2.4)-(2.7) that

$$D=k\sqrt{1-c}\,I+\frac{1}{2}\nu I^2,\ I_0=\frac{2}{5}(1-c)\frac{k_1}{\nu+k\sqrt{1-c/I}},\qquad(2.8)$$
$$I^2=I_0^2+\frac{2}{5}c(1-c)k_1^2/(\nu+k\sqrt{1-c/I})^2.$$

The latter two equations resolve the indeterminacy with respect to k, so the dissipative function takes the following form for rigid inclusions:

$$D=k\sqrt{1+\frac{3}{2}c}\,I_0+\frac{1}{2}\nu\frac{1+\frac{3}{2}c}{1-c}I_0^2.\qquad(2.9)$$

One can readily derive the plasticity and viscosity limits in (2.9) for the material containing rigid inclusions.

The critical value I_0^* of the strain intensity at which the stresses in the inclusions attain the plasticity limit k_1 is given by (2.8) as

$$k_1=\frac{5}{2}\left(\frac{k}{\sqrt{1+\frac{3}{2}c}}+\frac{\nu}{1-c}I_0^*\right).\qquad(2.10)$$

We transfer to determining the rheological model when both phases are in the plastic state. It follows from (2.10) that for $k_1\leqslant(5/2)k$ the inclusions are in the plastic state from the start of deformation. In other cases, (2.9) applies for rigid inclusions at the start, and then when I_0^* is attained the general relations (2.5)-(2.7) apply, which incorporate the inclusion plasticity.

One can represent the relationships between $\tau=\sqrt{\langle s_{ij}\rangle\langle s_{ij}\rangle}/k$ and $\gamma=I_0\nu/k$ for various c with the parameters of a composite in the (τ,γ) plane as a family of curves separated by the boundary $\gamma=\gamma^*(c)$, $\tau=\tau^*(c)$ (Fig. 2)

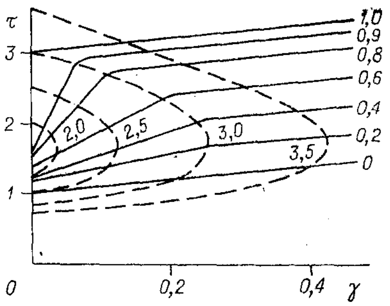

Fig. 2

$$\gamma^*(c) = (1-c)\left(\frac{2}{5}\frac{k_1}{k} - 1\Big/\sqrt{1+\frac{3}{2}c}\right), \quad \tau^*(c) = \frac{2}{5}\frac{k_1}{k}\left(1+\frac{3}{2}c\right),$$

with this corresponding to the points where the inclusions go over to the plastic state. These boundaries are shown by dashed lines indicating the values of k_1/k in Fig. 2. To the left of the boundary, the material is characterized by the Bingham relations:

$$\tau = \sqrt{1+\frac{3}{2}c} + \frac{1+\frac{3}{2}c}{1-c}\gamma,$$

and to the right by a nonlinear relationship, which for $\gamma \to \infty$ approximates a linear one with the asymptotic limits to the plasticity k_a and viscosity ν_a:

$$\nu_a/\nu = 1 - c\,(1-\nu_1/\nu)\Big/\left[1 - \frac{2}{5}(1-c)\left(1-\frac{\nu_1}{\nu}\right)\right],$$

$$\left(1-\frac{2}{5}(1-c)\left(1-\frac{\nu_1}{\nu}\right)\right)(k_a/k) = \left[(1-c)\sqrt{\left(1-\frac{2}{5}\left(1-\frac{\nu_1}{\nu}\right)\right)^2 + \frac{6}{25}c\left(1-\frac{\nu_1}{\nu}\right)^2 + \frac{k_1}{k}c}\right].$$

It is necessary to use these formulas because it is technically difficult to examine the microstructure at high temperatures. The formulas enable one to relate the parameters measured in macroscopic experiments to the concentrations and parameters of the components such as corundum and baddeleyite, which are widely used in refractory composites [4]. These components have been examined in some detail at various temperatures. However, theoretical methods have to be used to relate the structuring to the thermal history. The formulas can be used to calculate the parameters of a matrix containing ZrO_2 inclusions. Then the self-consistency hypothesis can be employed to calculate the properties of the matrix as a combination of corundum crystals and a vitreous bonding agent, i.e., one again applies the formulas derived here, but this time for rigid inclusions. The properties of ZrO_2 and of corundum crystals are stable, but the concentration of the vitreous phase is very much dependent on the thermal history, which has to be calculated. The macroscopic parameters of the material as functions of temperature can be calculated from experimental data, for example for torsion (Fig. 1). The data are closely fitted by (1.1) and (1.2), which are represented by theoretical curves in Fig. 1 (---) for $t^\circ = 1200^\circ C$ ($kR^3 = 4.78$, $\nu R^4 = 5.73\cdot10^4$, $k_aR^3 = 13.14$, $\nu_aR^4 = 0.38\cdot10^4$); $t^\circ = 1300^\circ C$ ($kR^3 = 2.39$, $\nu R^4 = 4.08\cdot10^4$, $k_aR^3 = 10.37$, $\nu_aR^4 = 0.25\cdot10^4$); $t^\circ = 1400^\circ C$ ($kR^3 = 1.91$, $\nu R^4 = 0.25\cdot10^4$, $k_aR^3 = 8.84$, $\nu_aR^4 = 0.16\cdot10^4$).

LITERATURE CITED

1. A. Auerbach, "Tests at 1600°C for establishing the mechanical qualities of certain electrically melted refractory materials," in: Proc. 7th Conf. Silicate 1963, Budapest, Hungarian Academy of Sciences (1965).
2. D. D. Ivlev, Theory of Ideal Plasticity [in Russian], Nauka, Moscow (1966).
3. V. V. Dudukalenko and N. N. Lysach, "The plastic parameters of a material containing platy inclusions," MTT, No. 1 (1980).
4. D. N. Poluboyarinov and G. Ya. Ponil'skii (eds.), Ceramics Made of Refractory Oxides [in Russian], Metallurgiya, Moscow (1977).